Modeling Time in Hybrid Systems: How Fast Is "Instantaneous"?

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Abstract

Many of today's electro-mechanical devices exhibit both continuous and discrete behavior. Modeling these *hybrid systems* presents special challenges for automated modeling and simulation. We show how nonstandard analysis overcomes these challenges, provides a firm mathematical foundation, and satisfies our intuitions about the behavior of hybrid systems.

1 Introduction

Many of today's electro-mechanical devices exhibit both continuous and discrete behavior. Modeling these *hybrid systems* presents special challenges for automated modeling and simulation. Work in discrete event simulation [2] assumes that all change is discrete; work in quantitative and qualitative simulation assumes that all change is (at least piecewise) continuous. The behavior of hybrid systems, such as digitally controlled copiers, chemical plants, automobiles, etc., is not appropriately characterized as either continuous or discrete.

A hybrid model of a system is almost always the result of an abstraction that simplifies analysis and the prediction of behavior. For example, we often view closing a switch as causing the voltage difference across the switch to become 0 in an instant; a level sensor in a reactor vessel causes a pump to shut off and a valve to close in an instant. In principle, it is possible to construct continuous models of these behaviors, but they are considerably more complicated. In practice, the use of discontinuous abstractions is both ubiquitous and necessary. The transient behavior of, for instance, control electronics is often irrelevant to the task of analyzing the overall system. Complex sequences of discrete actions are also possible, such as when an automobile ignition is turned on (relative to the vehicle's motion) or a camera's shutter is depressed.

A satisfactory model for hybrid systems must support:

- discrete actions occurring in the presence of continuous change;

- complex sequences of discrete actions;
- the abstraction that discrete actions are instantaneous.

We can refine the third criterion: it must not be possible to measure the *duration* of a discrete action with a continuous real-valued clock.

There have been several attempts to introduce discrete changes into a standard continuous model. Problems with the mathematical semantics arise, however, because discrete changes violate the assumption of continuity. Giving sound semantics to the representation of discrete changes while employing the real number line as the model of time (as is usually employed in modeling of continuous systems) and respecting the underlying semantics of continuous change turns out to be very difficult.

We provide a sound mathematical basis for modeling hybrid systems that satisfies the three desiderata listed above. The hybrid systems are specified by discrete actions as well as qualitative or quantitative continuous functions. Our solution is based on the calculus of hyperreals, i.e. nonstandard analysis [8]. We employ a nonstandard model of time, which captures the intuitive distinction we would like to make between discrete and continuous changes. More importantly, it allows us to model both continuous and discrete changes uniformly without contradictions or introducing unnecessary complexity.

2 Discrete Actions in Continuous Systems

Consider the simple circuit shown in Figure 1, in which electric power is provided to a load either by a solar array or a rechargeable battery. The charge on the battery is also maintained by a solar array. When the charge level of the battery exceeds a threshold, a charge-current controller opens a relay, allowing the battery to provide power to the load. When the charge level drops below another threshold, the charge-current controller closes the relay, allowing the solar array to recharge the battery. It is natural to model this system by a mixture of continuous and discrete behavior.

Figure 2 defines the continuous behavior of the system. The antecedents and consequences of C0–C3 hold simultaneously. If the antecedent holds at time t, then the consequence also holds at time t.

Figure 3 defines the discrete behaviors of the relay and controller. A discrete behavior is defined by a form $A : c \rightarrow e$, where A is the name of the discrete change, c is the condition for the change to take place, and e is its effect.

The discrete changes are extremely fast relative to the continuous action, and their dynamics are uninteresting for the purposes of modeling the overall circuit. Thus, we would like to model them as being *instantaneous*. Since each of them represents an actual action of a physical component, it does take some non-zero amount of time for the consequences to take effect after the condition becomes true. However, we would like

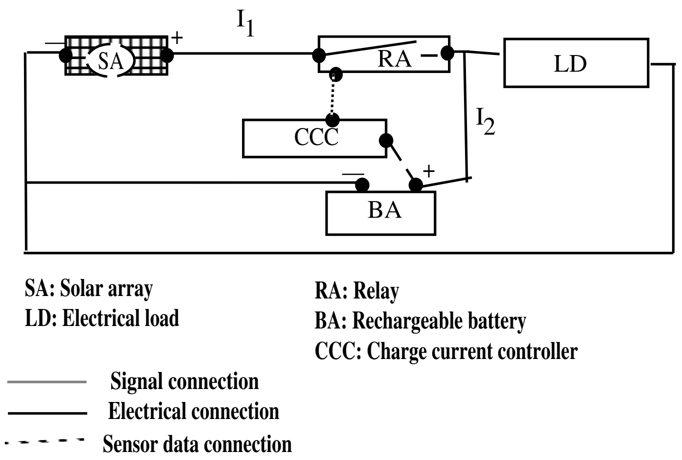

Figure 1: A simple circuit

C0: (Shining Sun) $\wedge$ (Closed Relay) $\Rightarrow I_1 = i_1$
When the sun is shining and the relay is closed, the solar array acts as a constant current source.
C1: (Closed Relay) $\Rightarrow I_1 + I_2 = 0$
When the relay is closed, it acts as a simple conductor.
C2: $\sim$(Closed Relay) $\Rightarrow I_1 = I_2 = 0$
When the relay is open, it conducts no current.
C3: $-I_2 = \frac{dQ_{BA}}{dt}$
The battery accumulates charge.

Figure 2: Axioms to describe the continuous behavior of the recharger circuit.

D1: (High Signal) $\wedge$ (Closed Relay) $\rightarrow$ $\sim$(Closed Relay)
When the signal from the controller goes high, the relay opens.
D2: $\sim$(High Signal) $\wedge$ $\sim$(Closed Relay) $\rightarrow$ (Closed Relay)
When the signal from the controller goes low, the relay closes.
D3: $Q_{BA} \geq q_2 \wedge$ $\sim$(High Signal) $\rightarrow$ (High Signal)
When the controller detects the charge level in the battery has reached q_2, it turns on the signal to the relay.

Figure 3: Discrete actions for the recharger circuit.

the model to capture the notion of *almost instantaneous change* taking place without any *measurable duration*.

While intuitively plausible, this interpretation of instantaneous changes raises a fundamental problem in modeling of continuous systems. Typically, time is taken to be isomorphic to the real number line. Thus, any temporal behavior can be viewed as a sequence of states that hold alternately over an instant and an open interval (the representation that is also used in qualitative reasoning). It works very well when there are no discrete changes. Without the discrete behaviors specified by D1 through D3, qualitative behavior of the systems may be something like what is shown in Figure 4. In the portion of the behavior shown in Figure 4, Q_{BA} is steadily increasing, until it reaches the threshold q_2 at time $t = t_2$. States s_0 and s_2 are instantaneous states at time points t_0 and t_2 respectively, and s_1 and s_3 are states corresponding to the open intervals $(t_1\ t_2)$ and $(t_2\ +\infty)$.

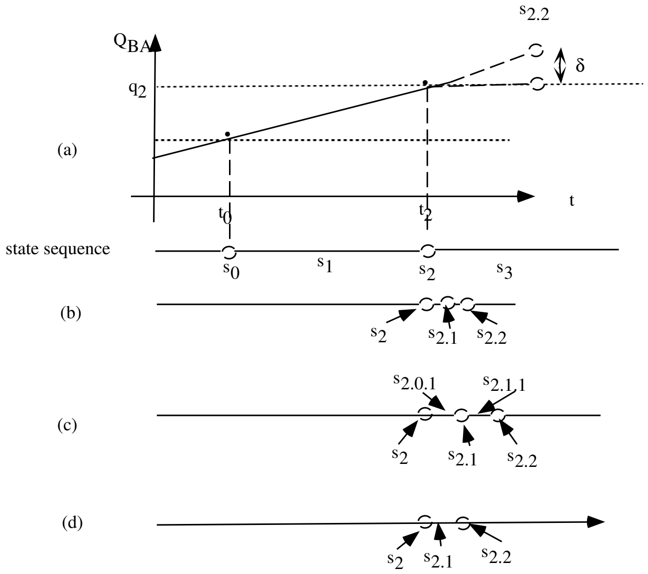

Figure 4: Behavior of the circuit in Figure 1.

Add to this behavior the actions of the controller and the relay represented by the specifications D1 through D3. The antecedent of D3 becomes true in the instantaneous state s_2. Thus, at the time $t_{2.1}$, immediately following t_2, the signal goes high. At yet another time ($t_{2.2}$) immediately following $t_{2.1}$, the relay opens due to action D1. Our intuitive notion about these instantaneous changes is that they happen so fast that the values of continuous variables do not measurably change during the short time required for the consequence of an action to take effect. This intended sequence of states are shown in Figure 4(b). The contents of the states are summarized below.

s_2 **at time** t_2 $Q_{BA} = q_2 \wedge$ (Closed Relay) $\wedge$ ~(High Signal)

The antecedent of D3 is true.

$s_{2.1}$ **at time** $t_{2.1}$ (High Signal) $\wedge$ (Closed Relay)
The consequence of D3 and the antecedent of D1 are true.

$s_{2.2}$ **at time** $t_{2.2}$ $\sim$(Closed Relay)
The Consequence of D1 is true.

If we use the real number line as the model of time, it is impossible to produce a description that matches exactly our intended interpretation of the discrete actions. On the real number line, there is no well-defined notion of a point immediately following a point. Even though we would like to say that there is a time point at which $Signal$ goes high and which "immediately follows" t_2, we cannot because the point t_2 must be followed by an open interval of non-zero length. This forces us to take one of the following approaches:

1. Since actions are supposed to take little time, assume that they take no time. In other words, rules such as D1 through D3 are treated just like ordinary logical implications with respect to time.

2. Always insert a small, open interval of unspecified length between the time points at which consequences of actions become true. This corresponds to the state sequence in Figure 4(c). States $s_{2.0.1}$ and $s_{2.1.1}$ represent the new open intervals.

3. Make the consequences of an action true in either the point or the interval that immediately follows the current state. This corresponds to the state sequence in Figure 4(d).

There are problems with all of these approaches. Option 1 is obviously problematic — if actions are taken as logical implication, then any of the rules D1 through D3 directly produces a contradiction. In general, there are many control actions that take place only if the desired effect is not already in place. The antecedent for such actions must include the negation of the consequence, and this will immediately lead to a contradiction if such rules are taken as logical implications.

With both Options 2 and 3, the value of "continuous" variables is not well defined after a sequence of actions. There are two possibilities shown in Figure 4(a). Since the sun remains up and the relay remains closed until state $s_{2.2}$, Q_{BA} continues to increase past q_2 until $s_{2.2}$. Since there is a non-zero amount of time that passes between s_2 and $s_{2.2}$, Q_{BA} must have some value over q_2, say $q_2 + \delta$, where δ is some positive quantity of unknown magnitude. As simulation continues and other discrete actions take place, variables can accumulate a number of such unknown δ's, unnecessarily complicating value computation.

If we ignore such δ's since we know that actions happen so fast that any change in the values of other continuous variables over the time is negligible, we introduce a contradiction. In the above example, if we assert that $Q_{BA}(s_2) = Q_{BA}(s_{2.2}) = q_2$, it will be inconsistent with the basic assumption that Q_{BA} is a continuous quantity and the fact that, in the given situation, the condition of C3 holds and therefore Q_{BA} should be continuously increasing over the interval between s_2 and $s_{2.2}$.

Option 3 arbitrarily assigns an instant or an open interval to the duration of an action depending on where it happens to appear in a sequence. If the action 1 in a sequence occurs at a time instant, then all odd numbered actions will occur at an instant. If it had occurred over an interval, then all of odd actions would occur at intervals. This is undesirable because it is a bizarre artifact of the particular model of time employed and has nothing to do with what the actions represent.

This example demonstrates problems which arise when one tries to represent hybrid systems while using the real number line as the model of time. The problems can be summarized as follows:

1. We cannot have a sequence of instantaneous states one immediately following the other.

2. We cannot ignore the change, if any, in the value of continuous variables over the time in which discrete actions take place.

We propose to use the hyperreals as our model of time. This allows us to represent discrete actions in a natural way and to overcome these two problems. The main advantages of using hyperreals are:

1. hyperreals allow us to have a sequence of time points one following the other such that the gap between the points are infinitely small.

2. hyperreals allow us to ignore any change in the value of continuous variables over a sequence of discrete actions as long the sequence is finite.

3 Calculus of Hyperreals

This section briefly reviews the fundamental concepts in nonstandard analysis that are relevant to our approach.

The calculus of hyperreals is a calculus defined over the set ${}^*R$, such that ${}^*R$ is a totally ordered field and ${}^*R$ contains R as its proper subfield. The members of R are called *standard members* of ${}^*R$. Nonstandard members of ${}^*R$ include infinite and infinitesimal numbers. The elements of ${}^*R$ can be generally classified as follows:

- An element ω of ${}^*R$ is called an *infinite hyperreal number* if $\forall a \in R,\ \omega > a$. We will denote the set of all infinite hyperreal numbers as ${}^*R_\infty$.

- An element ϵ of ${}^*R$ is called an *infinitesimal hyperreal number* if $\forall a \in R,\ |\epsilon| \leq |a|$. We will denote the set of all infinitesimal hyperreal numbers as ${}^*R_0$. Note 0 is the only standard member of ${}^*R_0$.

- An element b of ${}^*R$ is called a *finite hyperreal number* if $\exists$ a positive number $a \in R$, such that $|b| \leq a$. We will denote the set of all finite hyperreal numbers as ${}^*R_f$. Note ${}^*R_0 \subset {}^*R_f$.

We use lowercase alphabet letters to denote a member of ${}^*R_f$, ϵ with or without a subscript to denote a member of ${}^*R_0$, and ω with or without a subscript to denote a member of ${}^*R_\infty$.

The standard arithmetic operators are defined over ${}^*R$ in an intuitive manner. The following axioms follow from their definitions.

$$\epsilon_1 + \epsilon_2 = \epsilon_3$$

$$a \in {}^*R_f \Rightarrow a * \epsilon_1 = \epsilon_2$$

The value of $\omega * \epsilon$ can be a member of ${}^*R_f$, ${}^*R_\infty$, or ${}^*R_0$.

We will also use the notation $\approx$ to mean "infinitely close" defined as follows:

Definition 3.1: $a \approx b \equiv |a - b| = \epsilon$

The following theorem holds:

Theorem 3.2: *Each member of ${}^*R_f$ is infinitely close to a unique member of R.*

In other words, $\forall a \in {}^*R_f, \exists r \in R$ such that $a = r + \epsilon$ and r is unique. We will call such r the *standard part* of a and denote it as ${}^0a$.

Corollary 3.3: *Each interval of an infinitesimal length contains at most one element of R. Some examples are $(t - \epsilon,\ t + \epsilon)$, $(\epsilon, 2\epsilon)$, and $(\omega,\ \omega + \epsilon)$.*

In summary, a system ${}^*R$ of hyperreal numbers is R extended with infinite numbers of infinitesimal and infinite elements, and it is closed under addition and multiplication. A significant aspect of ${}^*R$ for our present purpose is that it gives us the notion of infinitesimal differences between two points of time (or quantity values) that are smaller than the difference between any two standard real numbers. Furthermore, infinitesimal differences never add up to a standard number as long as there are only a finite number of them.

In order to make ${}^*R$ our model of time (and the range of continuous functions), we must have a definition of continuity in ${}^*R$. In standard analysis, continuity of a function f at a is defined as

Definition 3.4: f is continuous in R iff $\forall e \exists d\, \forall x[|x - a| < d \rightarrow |f(x) - f(a)| < e]$.

In nonstandard analysis, continuity[1] of a function ${}^*f$ is defined in an analogous manner as:

[1]There are actually several different notions of continuity that can be defined in ${}^*R$. Q-continuity is one of them.

Definition 3.5: ${}^*f$ is Q-continuous in ${}^*R$ iff $\forall x[x \approx a \Rightarrow {}^*f(x) \approx {}^*f(a)]$.

The derivative ${}^*f'$ of ${}^*f$ is defined as follows:

Definition 3.6: If $\epsilon_1,\ \epsilon_2 \neq 0$ and $\epsilon_1,\ \epsilon_2 \in^* R_0$,

$$ {}^*f'(a) \equiv {}^0\left(\frac{{}^*f(a+\epsilon_1) -^* f(a)}{\epsilon_1}\right) $$

and

$$ {}^0\left(\frac{{}^*f(a+\epsilon_1) -^* f(a)}{\epsilon_1}\right) = {}^0\left(\frac{{}^*f(a+\epsilon_2) -^* f(a)}{\epsilon_2}\right). $$

In other words, the derivative is defined to be a standard number and the derivative is constant in the vicinity $(a-\epsilon,\ a+\epsilon)$ of a.

4 A Nonstandard Model of Hybrid Systems

We now describe our model of hybrid systems based on calculus of hyperreals. We will also show how the approach overcomes the difficulties discussed in Section 2. We use the hyperreals as the model of time as well as of the domain of continuous functions. We also assume that the functions used to describe the continuous part of the behavior are Q-continuous in ${}^*R$. As for discrete actions, we define the semantics of an action $A : c \rightarrow e$ as follows:

Definition 4.1: $A : c \rightarrow e$ means that $c(t_0) \Rightarrow \exists t_1$ such that $t_1 > t_0$ and $t_0 \approx t_1$ and $e(t_1)$.

In other words, when the antecedent of an action becomes true at time t_0, the consequence of the action becomes true at time t_1, which comes after t_0 but is infinitely close to t_0. Note that this definition itself does not require that e entail $\neg c$ (as is the case in all the examples of discrete actions in Figure 3). However, it is in general a good idea to represent actions in such a way that the consequence invalidates the condition because, otherwise, the action will end up being repeated an infinite number of times.

This definition of an action allows us to have a sequence of instantaneous states one after another, each of which is distinct but infinitely close to its predecessor. Furthermore, the value of a continuously changing variable changes only by an infinitely small magnitude over a finite sequence of such instantaneous states. Thus, in computing the standard part of the value of a continuously changing variable, we can always ignore the nonstandard part, because the nonstandard part can never become large enough to make a difference in the standard part.

In the example of the circuit in Figure 1, we will have a state sequence as depicted in Figure 4(c) except that the time line is now the hyperreal number line. The states $s_2,\ s_{2.1}$, and $s_{2.2}$ are distinct states, but the gaps between them are of infinitesimal magnitudes. Thus, we can safely say that the standard part of the value of Q_{BA} in state $s_{2.2}$ is equal to that in state s_2 without contradicting the continuity assumption or the equation in C3.

Notice that this semantics of continuous and discrete behavior based on nonstandard model of time allows us to capture in the most natural way what we mean intuitively by discrete actions without violating the basic continuity assumptions. It also allows us to avoid introducing δ's of an unknown magnitude into the value of continuously changing variables, unnecessarily complicating computation.

4.1 Temporal Projection

The task of modeling hybrid systems require both a mathematical foundation that allows the behavior of a hybrid systems to be described and algorithms that predict the behavior from such a description. In this section, we briefly discuss the problem of prediction, particularly with regard to predicting behavior across discrete changes.

Predicting behavior requires us to solve the "temporal projection" problem. During phases of continuous behavior, temporal projection is straightforwardly solved by differential calculus. The equations describing a system together with the values of variables can be solved to determine future behavior. Difficulties may arise when a discrete action occurs since a single discrete change may cascade, resulting in many discontinuities. For instance, dumping hot water into a container holding some cold water results in discontinuous changes in the mass of water, its level, temperature, heat, pressure at the bottom, and so on. It does not, however, change the location of the pot, its color, and so on. As Forbus points out in [6], there is no formal standard for correctness here, there are only informal desiderata. The primary one is that changes should be minimal and causally related to the action. The problem of retaining predications across an action has been widely studied in the AI literature. There are two basic approaches: either explicit frame axioms are required to carry predications across discrete changes (STRIPS [5]), or the logic is extended with some sort of accessibility relation and preference relation between possible worlds (AE [6], any of the non-monotonic logics for expressing action). Unfortunately, neither approach is altogether satisfactory. Providing explicit frame axioms is error-prone and difficult because the frame axioms cannot be specified for individual actions or predicates in isolation. Providing an accessibility and preference relation that eliminates implausible consequences (but not plausible ones) while being computationally tractable remains elusive. Combinations of the two approaches appear to be quite promising.

The Device Modeling Environment (DME) [9] combines explicit frame axioms with a preference relation. DME uses an algorithm for temporal projection over discrete changes that appears reasonably efficiently and avoids implausible consequences. DME is a modeling and simulation program for hybrid systems where continuous changes are described by a set of algebraic and ordinary time differential equations and discrete changes are described by actions as discussed throughout this paper. When a discrete change takes place, DME prefers among all the states that can result from the action those states that satisfy the following criteria:

1. The consequence of the action is true in the state.
2. The values of a variable that is exogenous, integrated, or discrete remains the same unless the variable is explicitly changed by the action.
3. The values of the variables that are specified not to change across discrete changes in user-provided, domain-specific frame axioms remain the same.

Integrated variables are those quantities whose values at time t is the integration of changes up to that time; unless changed explicitly, their values should not change instantaneously. Likewise, exogenous variables are those controlled by entities external to the model; unless changed explicitly, their values are not likely to change. Finally, since DME assumes that all mechanisms for change (continuous or discrete) are represented as equations or actions, and continuous equations cannot change the value of a discrete variable, the value of such a variable is likely to remain the same unless changed explicitly by an action.

This strategy avoids producing inconsistencies by being conservative about value projection while allowing for efficiency improvement when domain-specific frame axioms are available.

5 A Logic for Hybrid Systems

Section 4 has defined a model of hybrid systems based on nonstandard analysis that satisfies the desiderata outline in the introduction. This model may be employed in several ways. It may be embedded into first order logic. A common method for representing actions and change in first order logic is to take time-varying predicates and augment them with an additional argument that ranges over the times that the predicate holds. This argument may be allowed to range over the hyperreals instead of the reals. If the mathematical definitions of continuity, etc., over the hyperreals are added, then one can reason about the behavior of systems so described. It is often desirable, however, to construct a slightly restricted logic that will enforce the common idioms and allow them to be more succinctly expressed. For example, temporal logics typically prevent explicit reference to and quantification over time, as well as making temporal statements much more succinct. It is possible to define a temporal logic similar Henzinger's HTL [1], and replace the real line with hyperreals. We will pursue a third possibility here and construct a logic specifically for the purpose of predicting and analyzing the behavior of hybrid systems. The key idea is that the denotation of sentences will be given by possible temporal behaviors where time can take on hyperreal values.

Our logic is based on the approach of concurrent constraint programming. Concurrent constraint programming uses the idea of a *store* as the set of possible values of the variables. Programs can then add constraints to the store, and ask the store if some constraints are valid.

Our language modifies the standard concurrent constraint languages, which are atemporal, by allowing the language constructs to extend across time. Thus, the store also varies over time. The language is built over a constraint system, and we assume that the constraint system is powerful enough to express the desired properties. In particular, the constraint system can express differential equations and propositional logic.

The syntax of the language is:

$$A ::= c \mid c \Rightarrow A \mid c \rightarrow_{\epsilon} A \mid A \| A \mid \textbf{first}\ c$$

c represents the constraint being added to the store. We will assume that it stays there until a discrete action adds its negation to the store. $c \Rightarrow A$ is used to represent simultaneous actions, for example those in Figure 2. $c \rightarrow_{\epsilon} A$ represents a discrete action, so A holds an ϵ time after c becomes true. $A \| B$ is used to put together several such constructs to form a program. **first** c is used to specify that c is true at the start of an interval.

The model we have for these programs is a set of functions from the hyperreals to sets of constraints. Each such function represents a possible evolution of a program, describing the constraints that are present in the store at any time instant. The only restriction that we place on these sets of functions is that they be determinate i.e. for any evolution o up to time t, the set $\{f(t) \mid f \text{ extends } o\}$ is closed under greatest lower bounds.[2] This enables us to determine uniquely the output of a process given as a set of functions.

The denotation of a program P in our language, written $[\![P]\!]$, is the set of all of its possible evolutions. We can define the denotation compositionally as follows. $[\![c]\!]$ is the set of all functions where $f(t) \supseteq c$ until some t when $f(t) \supseteq \neg c$. $[\![c \Rightarrow A]\!]$ is the set of all functions in which whenever c is true at time t, then the function starting at t is in $[\![A]\!]$. $[\![c \rightarrow_{\epsilon} A]\!]$ contains those functions f in which whenever c is true at t, then f starting at $t + \epsilon$ is in $[\![A]\!]$. Notice that in order to get determinacy, we need to know a fixed ϵ, and also since ϵ is an infinitesimal, we need hyperreal functions here. $[\![A \| B]\!] = [\![A]\!] \cap [\![B]\!]$. This last definition provides the motivation for including all of the functions $f(t) \supseteq c$, rather than something like $f(t) = c$. Including the supersets allows composition to be defined as intersection.

As an example, consider the following program:

$$\begin{aligned}&x' = 1 \| \textbf{first}(x = 0) \| \text{power\_on} \| \\ &x = 3 \rightarrow_{\epsilon_1} \text{relay\_open} \| \text{relay\_open} \rightarrow_{\epsilon_2} \text{power\_off} \| \text{power\_off} \rightarrow_{\epsilon_3} x' = 0.\end{aligned}$$

For any function f in its denotation, we must have $f(0) \supseteq \{x' = 1, x = 0, \text{power\_on}\}$, $f(3) \supseteq \{x' = 1, \text{power\_on}\}$, $f(3 + \epsilon_1) \supseteq \{x' = 1, \text{power\_on}, \text{relay\_open}\}$, $f(3 + \epsilon_1 + \epsilon_2) \supseteq$

[2] f extends o if $f(x) = o(x)$ for all $x < t$, where o is defined up to t.

$\{x' = 1, \text{power\_off}, \text{relay\_open}\}$, and so on. This gives us all the information we need about the process, and we can use it to deduce various things as shown below.

Once we have a denotational model for our programs, we immediately get a logic for the language. Given programs A, B we say $A \vdash B$ if $[\![A]\!] \subseteq [\![B]\!]$, that is every possible evolution of A is a possible evolution of B. We then build up an inference system for this logic, see [14] for details. We can use this logic to reason about programs. For example, if B is known never to get into a bad state, and $A \vdash B$ then we know that A can never get into a bad state. Thus, in the above example we can prove that $P \vdash x \leq 3$, which is a desired safety property.

The language described here is, of course, not a full-fledged modeling language. It does not provide a succinct way of characterizing temporal evolution using defaults such as TCC [14], nor does it provide a succinct syntax for describing physical systems and the processes that effect them such as the CML [4]. However, it does illustrate the basic ideas described in this paper, and we will develop it into a full language in a separate paper.

6 Related Work

There has been a considerable amount of work that addresses the problems of reasoning about hybrid systems or has looked at the issues of using nonstandard analysis to represent processes acting at different orders of magnitude.

Rayner [13] suggested use of nonstandard analysis to model continuous systems with discrete changes in his defense of classical logic as means of modeling continuous system behavior.

Henzinger's hybrid temporal logic [1] allows the behavior of piecewise-continuous systems to be described and enables properties of these behaviors to be verified (by hand). This work uses a real model of time together with limits to describe discontinuities. HTL does not allow for a sequence of actions. The work has been mostly "descriptive", rather than "predictive".

Forbus introduced the notion of an "action augmented envisionment" [6] that incorporates discrete instantaneous actions into his Qualitative Process theory [7]. It appears likely that this approach is consistent with the representation that we have described. It is difficult to be certain, because there is no commitment to a model of time. There are several limitations of Forbus' approach. First, only a single action may occur at a time. Forbus observes that this is not a fundamental limitation, as compound actions may be defined. This is, however, an important practical limitation — it makes it impossible to define any action in isolation of others. This may seem palatable when considering actions taken by a single agent, but when there are multiple agents it becomes problematic. Second, there cannot be any sequences of actions. Third, actions can only change the truth of atomic ground formulae (the STRIPS action model). This means that actions cannot introduce new objects into the system depending on its state. Fourth, the algorithms presented to infer the behavior of a system to which actions might be applied do not scale. They effectively apply each action whenever it can be applied to all possible states that the system might ever be in. Forbus suggests that incremental algorithms should be possible, but they are not further developed. One can view the algorithms described in this paper and implemented in the DME system as incremental algorithms for achieving this purpose. Finally, the state that results from applying an action is determined heuristically. The state that results from applying an action is the state that is consistent with the action and most like the one in which the action was applied. In his implementation, "most like" means sharing the maximal number of assumptions. There is no place in the representation for explicitly stating frame axioms, they are all implicitly defined by the "nearest neighbor" heuristic. Forbus observes that there is no formal standard for correctness — only an informal set of criteria that should be satisfied. In particular, an action should result in no extraneous changes and that only the minimal necessary changes should be predicted.

Nishida and Doshita proposed two methods, called approximation and direct methods, to handle discontinuous changes in simulating the behavior of a mostly continuous system [11]. The approximation method models a discontinuous jump in a continuous variable value as a gradual change and carries out envisionment of the behavior during the gradual change using infinitesimals. The approximation method works well when discontinuous change is in an input variable value, and the variable is a continuous-valued variable. However, Nishida and Doshita state that it is not clear how well the method will perform in other cases where a discontinuous change is caused by a mode transition, positive feedback without time delay, or a change in the value of a discrete-valued variable.

The direct method predicts a sequence of mythical instantaneous states between normal states when a discontinuous change takes place. The mythical instances are states where the variables do not satisfy all the system constraints; the method produces a series of them as it searches for a consistent state by sequentially relaxing assumptions that cause inconsistencies. This method seems to predict correctly the consequences of discrete changes while producing a causal account of what happens when such discrete changes take place for any types of discrete changes. De Kleer and Brown also uses the notion of mythical states to produce a causal account of how disturbances propagate through a model to cause a change, though they do not handle discrete changes [3]. The problem with the notion of "mythical" states in both cases is that it is not clear what they actually represent. In other words, it is not clear whether mythical states represent very short but real instances or they are artifact of the representation and reasoning procedures. If they do represent real instances, the semantics of the underlying model of time becomes unclear.

Raiman used nonstandard analysis as the basis for his theory of order of magnitude reasoning [12]. His work on order of magnitude reasoning is totally within the realm of continuous systems. Even though we believe that some types of discontinuous changes can be modeled as continuous changes using order of magnitude reasoning, as Nishida and Doshita showed, other types of discontinuous changes such as changes in symbolic variables, do not lend themselves easily to this approach.

Tanaka and Tsumoto [15] present a qualitative calculus employing ranked hyperreals that is quite similar to the one presented here. They show how it can be used to do qualitative order-of-magnitude as well as time-scale analysis. Unfortunately, the analysis and algorithms presented remain at the qualitative symbolic level. The semantics of the system is not defined in terms of the properties of functions on ranked hyperreals (such as continuity). They do not discuss the use of ranked hyperreals to model discrete changes, but limit its use to describing physical systems that are described as continuous down to the lowest level of detail. The system is an extension of Kuipers' time scale abstraction [10], which allows a system to be decomposed and each time scale simulated independently.

7 Conclusion

While hybrid systems have become evermore commonplace, analysis methods have failed to keep pace and have focussed on either (piecewise) continuous or discrete systems. A contributing factor has been the lack of an adequate model for the behavior of hybrid systems. We have shown that approaches in which time is modeled by the real number line fail to satisfy key desiderata. Fortunately, we have also shown that an approach in which time is modeled by the hyperreal line can satisfy these desiderata. Our model for hybrid systems supports:

- discrete actions occurring in the presence of continuous change. Continuity is well defined on the hyperreal line and the standard part of a continuous function's value is unchanged across any infinitesimal interval. Thus, values can be projected across actions without introducing any contradictions.
- complex sequences of discrete actions. Arbitrary finite sequences of actions may occur in our model.
- the abstraction that discrete actions are instantaneous. A real valued continuous clock cannot measure the infinitesimal duration of a sequence of actions.

Furthermore, our model allows actions to take different amounts of time before their consequences take effect (e.g. one action can be twice as fast as another).

We have used our model in two ways: to provide a semantics for DME's behavioral predictions for hybrid systems, and to define a simple logic for the prediction and analysis of behavior. We are working to extend the logic to support defaults and the properties necessary to succinctly solve the temporal projection problem. This will enable us to provide a clean compositional semantics for rich device modeling languages such as CML and, with appropriate computational support, allow for properties of hybrid systems to be verified.

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